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On the Pesin set of expansive geodesic flows in manifolds with no conjugate points

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Abstract. In this paper, we show that the Pesin set of an expansive geodesic flow in compact manifold with no conjugate points and bounded asymptote coincides a.e with an open and dense set of the unit tangent bundle. We also show that the set of hyperbolic periodic orbits is dense in the unit tangent bundle.

Keywords: expansive geodesic flow, bounded asymptote, Pesin set.

Mathematical subject classification: 37D25, 37D40.

Introduction

The purpose of this paper is to study the Pesin set of expansive geodesic flows in manifolds without conjugate points.

Recall that the Pesin set of a C^∞ flow $\varphi_t : \mathcal{M} \to \mathcal{M}$ without singularities acting on a differentiable Riemannian manifold $\mathcal{M}$ is the set Λ of all points in $\mathcal{M}$ that satisfy the following property: For all $p \in \Lambda$ there exists a subspace $\hat{\Lambda}(p) \subset T_p\mathcal{M}$ which is transversal to the flow where the Lyapunov exponent of every vector is nonzero.

Definition. *Let $\varphi_t : \mathcal{M} \to \mathcal{M}$ be a continuous flow acting on a metric space $(\mathcal{M}, d)$. The flow φ_t is said to be expansive if there exists a constant $\varepsilon > 0$ such that for every $p \in \mathcal{M}$, we have the following property: If for a given $q \in \mathcal{M}$ there exists a continuous surjective map $\rho : \mathbb{R} \to \mathbb{R}$, with $\rho(0) = 0$ such that*

$$d(\varphi_t(p), \varphi_{\rho(t)}(q)) \leq \varepsilon$$

for every $t \in \mathbb{R}$, then there exists $t_0 \in \mathbb{R}$ such that $\varphi_{t_0}(p) = q$.

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In other words, two different orbits of an expansive flow become eventually separated by a distance of at least ε. Geodesic flows of compact, Riemannian manifolds with negative curvature are examples of expansive flows.

Recall that a Riemannian manifold $\mathcal{M}$ is said to have no conjugate points if, the exponential map at every point is non-singular.

The results in our paper are motivated, by the theory of rank one on manifolds of nonpositive curvature (expansive geodesic flows in such manifolds clearly imply rank one) and by the following result due to G. Knieper [10]: If a compact manifold without conjugate points and bounded asymptote has zero metric entropy, then it is a torus.

We say that a manifold with no conjugate points has bounded asymptote if there exists a positive constant C such that for any unit speed geodesic γ in $\mathcal{M}$ and any stable Jacobi vector field J on γ, $\| J(t) \| \leq C \| J(0) \| \; \forall t \geq 0$ (For the definition of stable Jacobi field, see section 1).

Therefore by Knieper's result, a manifold $\mathcal{M}$ with no conjugate points, bounded asymptote and whose universal covering $\hat{\mathcal{M}}$ is a Visibility manifold (or a Gromov hyperbolicity space, see Ruggiero [16]) has positive metric entropy (for the definition of topological and metric entropy see for instance Mañe [12]). By a result of Ruggiero [16], the topological entropy of an expansive geodesic flow in a compact manifold with no conjugate points is positive and the universal covering of the manifold is a Visibility manifold. This result, together with Knieper's theorem imply that expansive geodesic flows in compact manifolds with no conjugate points and bounded asymptote, have positive metric entropy. The main result of the paper is that in fact, the Pesin set of such flows has positive measure.

Theorem A. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points, bounded asymptote and whose geodesic flow is expansive. Then the Pesin set of the geodesic flow coincides a.e with an open and dense set of the unit tangent bundle. Moreover, the set of closed hyperbolic geodesics is dense in the unit tangent bundle.*

We would like to point out that the Pesin set of the geodesic flow of a compact rank one manifold of nonpositive curvature coincides a.e with an open and dense set. Theorem A extends this feature to the family of expansive geodesic flows in compact manifolds with bounded asymptote.

It is interesting to notice that many important convexity results of the theory of rank one manifold of nonpositive curvature, like the Flat strip theorem, might not hold just assuming bounded asymptote (Burns [4]).

1 Preliminaries

In this section we recall some basic notions. Throughout the paper, $\mathcal{M}$ denotes a C^∞ compact Riemannian manifold of dimension $n \geq 2$ with no conjugate points. The tangent and unit tangent bundles of $\mathcal{M}$ are denoted by $\mathcal{T}\mathcal{M}$ and $S\mathcal{M}$ respectively. Let $\pi : S\mathcal{M} \to \mathcal{M}$ be the natural projection on $\mathcal{M}$ and $\hat{\pi} : \hat{\mathcal{M}} \to \mathcal{M}$ its universal covering. For any $v \in \mathcal{T}\mathcal{M}$ denote by $\mathcal{K} : T_v\mathcal{T}\mathcal{M} \to T_{\pi(v)}\mathcal{M}$ the connection map. Using the natural projection and the connection map, we obtain a Riemannian metric on $\mathcal{T}\mathcal{M}$ called the Sakaki metric. That is, let $v \in \mathcal{T}\mathcal{M}$ and $\zeta, \eta \in T_v\mathcal{T}\mathcal{M}$, then

$$\langle \zeta, \eta \rangle_v = \langle d\pi\,\zeta, d\pi\,\eta \rangle_{\pi(v)} + \langle \mathcal{K}(\zeta), \mathcal{K}(\eta) \rangle_{\pi(v)}.$$

Let $v = (p, \theta) \in S\mathcal{M}$, we shall denote by γ_v the unit speed geodesic on $\mathcal{M}$ such that $\gamma_v = p$ and $\gamma_v'(0) = \theta$. Let $\mathcal{J}(v)$ be the $2n$-dimensional vector space of Jacobi vector fields on γ_v. If $w \in T_{\pi(v)}\mathcal{M}$, let $J_{w,t} \in \mathcal{J}(v)$ be the unique Jacobi vector field on γ_v such that $J_{w,t}(0) = w$ and $J_{w,t}(t) = 0$. If w is perpendicular to v, Green [7] proves that $\lim_{t\to\infty} J_{w,t}$ and $\lim_{t\to-\infty} J_{w,t}$ always exist and these limits are Jacobi vector fields on γ_v.

For all $v \in S\mathcal{M}$, let $\mathcal{J}^s(v)$ and $\mathcal{J}^u(v)$ be respectively, the set of all perpendicular Jacobi vector fields on γ_v such that $J_{w,t} \to J$ as $t \to \infty$, $t \to -\infty$ where $J_{w,t}$ is defined above. Let $\mathcal{J}^c(v) = \mathcal{J}^s(v) \cap \mathcal{J}^u(v)$.

Definition 1.1. *Let $v \in S\mathcal{M}$, a Jacobi vector field J on γ_v is called stable if $J \in \mathcal{J}^s(v)$, J is called unstable if $J \in \mathcal{J}^u(v)$ and J is called central if $J \in \mathcal{J}^c(v)$.*

Let $v \in S\mathcal{M}$, let $G(v)$ be the (unit) vector tangent to the geodesic flow and let N_v be the subspace of $T_vS\mathcal{M}$ that is normal to $G(v)$ in the Sakaki metric.

In fact, letting $H_v = Ker(\mathcal{K})$ and $V_v = Ker(d\pi)$ we have that $T_vS\mathcal{M} = H_v \oplus V_v \oplus G(v)$. H_v, V_v and $G(v)$ are mutually orthogonal in the Sakaki metric, so each $\zeta \in N_v$ can be expressed in coordinates by $\zeta = (d\pi(\zeta), \mathcal{K}(\zeta))$. Let J_ζ be the Jacobi vector field defined on γ_v whose initial conditions are $J_\zeta(0) = d\pi\,\zeta$ and $J_\zeta'(0) = \mathcal{K}(\zeta)$. This Jacobi field is unique and the map $\zeta \to (J_\zeta(0), J_\zeta'(0))$ commutes with the action of $d\varphi_t$, that is, $d\varphi_t\zeta = (J_\zeta(t), J_\zeta'(t))$ for all $t \in \mathbb{R}$.

The images in $T_vS\mathcal{M}$ of the sets $\mathcal{J}^s(v)$, $\mathcal{J}^u(v)$ and $\mathcal{J}^c(v)$ under the above isomorphism will be denoted by $X^s(v)$, $X^u(v)$ and $X^c(v)$ respectively.

The following proposition can be found in Eberlein [6].

Proposition 1.1. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points. Let $v \in S\mathcal{M}$ and $k > 0$ such that $-k^2$ is a lower bound of the sectional curvature. Then*

1. *For any* $\zeta \in X^s(v)$ *or* $X^u(v)$ *we have that* $\| \mathcal{K}(\zeta) \| \leq k \| \mathrm{d}\pi\zeta \|$.
2. *Let* $\zeta \in T_v S\mathcal{M}$ *such that* $\langle \zeta, G(v) \rangle = 0$ *and* $\| \mathrm{d}\pi \mathrm{d}\varphi_t \zeta \|$ *is bounded above for all* $t \geq 0$ *(respectively for all* $t \leq 0$*). Then* $\zeta \in X^s(v)$ *(respectively* $\zeta \in X^u(v)$*).*

2 Bounded Asymptote and Regular vectors

Definition 2.1. *$\mathcal{M}$ satisfies the bounded asymptote condition if there is a positive constant C such that for all $v \in S\mathcal{M}$ and $J \in \mathcal{J}^s(v)$,*

$$\| J(t) \| \leq C \| J(0) \| \qquad \forall t \geq 0.$$

Note that by 1 of proposition 1.1, $\mathcal{M}$ satisfies the bounded asymptote condition if and only if there is a positive constant C such that for all $\zeta \in X^s(v)$,

$$\| d\varphi_t \zeta \| \leq C \| \zeta \| \qquad \forall t \geq 0.$$

Manifolds with no focal points satisfy the bounded asymptote condition. Gulliver [8] gives an example of a compact manifold with no focal points whose sectional curvature change sign.

If $\mathcal{M}$ has bounded asymptote, based on the notion of rank in manifolds of nonpositive curvature, we define the rank of v ($rank(v)$) as the dimension of the vector space $\mathcal{J}^c(v)$ plus one. The vector v is called regular if there exists an open neighborhood U of v such that for all $w \in U$, $rank(w) = rank(v)$. The set of all regular vectors will be denoted by $\mathcal{R}$.

For a reference of the following result see Eberlein [6] for instance.

Lemma 2.1. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and bounded asymptote.*

1. *The sets*

$$A^s = \bigcup_{v \in SM} X^s(v) \quad \textit{and} \quad A^u = \bigcup_{v \in SM} X^u(v)$$

are closed subbundles of $T(S\mathcal{M})$, that is, $v \to X^s(v)$ and $v \to X^u(v)$ are continuous maps.

2. *The map $v \to rank(v)$ is an upper semicontinuous function.*

The following result is straightforward from the fact that the rank is an upper semicontinuous entire function, its proof is completely analogous to the case of nonpositive curvature (see for instance Ballmann [2]).

Lemma 2.2. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and with bounded asymptote. The set $\mathcal{R}$ of all regular vectors is an invariant, dense set in $S\mathcal{M}$.*

Proof. By definition $\mathcal{R}$ is invariant. Also $\mathcal{R}$ is the set of continuity points of $v \to rank(v)$. Since $v \to rank(v)$ is upper semicontinuous, its set of continuity points is generic, in particular, dense. □

Next we state a basic lemma.

Lemma 2.3. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and with bounded asymptote. Let $v \in S\mathcal{M}$ a regular vector such that $rank(v) \geq 2$. Then there exists a continuous vector field $\mathcal{F}$ defined on an open neighborhood U of v such that $\mathcal{F}(w) \in X^c(w)$ for all $w \in U$.*

Proof. We give a sketch of proof for the sake of completeness. By the bounded asymptote condition, there exists a neighborhood $V \subset \mathcal{R}$ of v such that $X^c(w)$ depends continuously on w for all $w \in V$. Without loss of generality we can suppose that V is a convex neighborhood of v. Let $\zeta \in X^c(v)$ a fixed vector. For any $w \in V$, let P_{vw} be the parallel transport of ζ along the unique geodesic joining v, w and let $\mathcal{F}(w)$ be the orthogonal projection of P_{vw} on $X^c(w)$.

By continuity there exists an open neighborhood of $W \subset V$ v such that $\mathcal{F}(w)$ is non-trivial for all $w \in W$. Note that by construction, $\mathcal{F}$ is a continuous vector field on W. □

The following result would be a straightforward application of the Flat strip theorem if we assumed nonpositive curvature. However if we allow focal points, the flat strip theorem is no longer true (see Burns [4]).

Lemma 2.4. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and with bounded asymptote. If the geodesic flow is expansive, then for all $v \in \mathcal{R}$, $rank(v) = 1$.*

Proof. For otherwise, there would exist a regular vector $v \in S\hat{\mathcal{M}}$ such that $rank(v) \geq 2$. By lemma 2.3 there exists an open neighborhood U of v and a continuous vector field $\mathcal{F}$ such that for all $w \in U$, $\mathcal{F}(w) \in X^c(w)$. By the theorem of existence and uniqueness of differential equations, there exists a

$\delta > 0$ an a curve α such that $\alpha : (-\delta, \delta) \to \hat{U}$, with $\alpha(0) = v$ and $\mathcal{F}(\alpha(s)) = \alpha'(s), \quad s \in (-\delta, \delta)$. We obtain a geodesic variation $\alpha_s(t) = \pi(\varphi_t(\alpha(s)))$ (where π is the canonical projection of $S\hat{\mathcal{M}}$ on $\hat{\mathcal{M}}$) whose variational fields $J_s(t) = \frac{\partial}{\partial(s)}(\pi\varphi_t(\alpha(s)))$, are central Jacobi vector fields. So, if $s_1 < s_2$ we have that

$$d(\alpha_{s_2}(t), \alpha_{s_1}(t)) \leq \int_{s_1}^{s_2} \| \frac{\partial}{\partial s}\alpha_s(t) \| \; ds = \int_{s_1}^{s_2} (\| J_s(t) \|^2 + \| J_s'(t) \|^2)^{1/2} ds$$

By bounded asymptote and proposition 1.1, we conclude that

$$d(\alpha_{s_2}(t), \alpha_{s_1}(t)) \leq \int_{s_1}^{s_2} (C^2 \| J_s(0) \|^2 + k^2 \| J_s(0) \|^2)^{1/2} ds \quad \forall t \in \mathbb{R}$$

where $-k^2$ is a lower bound of the sectional curvature of $\mathcal{M}$.

Therefore, there exists a positive constant $\hat{\mathrm{C}}$ such that

$$d(\alpha_{s_2}(t), \alpha_{s_1}(t)) \leq \hat{C}l(\pi(\alpha(s))), \qquad s \in (s_1, s_2) \quad \forall t \in \mathbb{R}.$$

Hence, α_{s_1} and α_{s_2} are two different asymptotic geodesics which contradicts the expansivity of the geodesic flow if we take δ small enough. □

Corollary 2.1. *Let $\mathcal{M}$ be a compact Riemannian manifold without conjugate points and bounded asymptote. If the geodesic flow is expansive, we have that $v \in S\mathcal{M}$ is a regular vector if and only if $rank(v) = 1$.*

Proof. If v is a regular vector then $rank(v) = 1$ by lemma 2.4.

Conversely, suppose that v is a non-regular vector, then there exists a sequence $w_n \to v$ such that $rank(w_n) \neq rank(v)$. As the rank function defines an upper semicontinuous entire function, it follows that $rank(w_n) < rank(v)$ and then, $rank(v) > 1$, but this is a contradiction. □

3 Pesin Set: Lyapunov Exponents

Let us recall the notion of Lyapunov exponent.

Definition 3.1. *Let $v \in S\mathcal{M}$. Let $\zeta \in T_v S\mathcal{M}$. The Lyapunov exponent $\mathcal{X}_v(\zeta)$ is defined by:*

$$\mathcal{X}_v(\zeta) = \lim_{t\to\infty} \frac{1}{t} log \| \mathrm{d}\varphi_t \zeta \| .$$

According to the definition of the Pesin set given in the introduction, the Pesin set of the geodesic flow is the set Λ of all points in $S\mathcal{M}$ that satisfy the following property: For all $v \in \Lambda$ there exists a codimension one subspace N_v of $TS\mathcal{M}$ that is orthogonal to the geodesic field $G(v)$ and such $\mathcal{X}_v(\zeta) \neq 0$ for all $\zeta \in N_v$.

In order to prove theorem A, we first study the behaviour of the norm of the stable and unstable vectors.

Lemma 3.1. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and bounded asymptote. Let $v \in S\mathcal{M}$ a rank one, regular, recurrent vector and $\zeta \in X^s(v)$, then*

$$\lim_{t\to\infty} \| d\varphi_t \zeta \| = 0.$$

Proof. By hypothesis, there exists a neighborhood $U \subset \mathcal{R}$ of v and a sequence of real numbers t_n such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \varphi_{t_n} v = v$.

We can assume without loss of generality that for all n, $\varphi_{t_n} v \in U$.

Claim:

$$\lim_{n\to\infty} \| d\varphi_{t_n} \zeta \| = 0 \text{ for all } \zeta \in T_v S\mathcal{M}.$$

Otherwise, there would exist $\zeta \in X^s(v)$ and $\epsilon > 0$ such that $\| d\varphi_{t_n} \zeta \| \geq \epsilon\ \forall n$.

Let $\zeta_n = d\varphi_{t_n} \zeta$. Then $\zeta_n \in X^s(\varphi_{t_n}(v))$ and

$$\| \zeta_n \| \geq \epsilon. \tag{1}$$

For every $r \geq -t_n$, by bounded asymptote,

$$\| d\varphi_r \zeta_n \| \leq C \| d\varphi_{-t_n} \zeta_n \| = C \| \zeta \| . \tag{2}$$

Also by bounded asymptote, the distribution X^s is continuous. Then $\zeta_n \to \hat{\zeta}$, where $\hat{\zeta}$ is a non-trivial stable vector in $X^s(v)$ (equation 1). From equation 2 we have that

$$\| d\varphi_t \hat{\zeta} \| \leq C \| \zeta \| \quad \forall t \in \mathbb{R}.$$

That is, $\hat{\zeta}$ is a non-trivial central vector. But this is a contradiction because $rank(v) = 1$ (central vectors fields do not exist in $X^s(v)$).

It follows that

$$\lim_{n\to\infty} \| d\varphi_{t_n} \zeta \| = 0 \quad \forall \zeta \in X^s(v).$$

To show lemma 3.1, we have to show that $\forall\epsilon > 0\ \exists T = T(\epsilon);\ \forall t > T$ $\| d\varphi_t\zeta \| \leq \epsilon$. This follows at once from the Claim and the fact that

$$\| d\varphi_t\zeta \| \leq C \| d\varphi_{t_n}\zeta \| \qquad \forall t \geq t_n \qquad \square$$

Applying lemma 3.1 to a basis $\zeta_1,..., \zeta_{n-1}$ for $X^s(v)$, we conclude that if v is a regular and recurrent vector, then

$$\lim_{t\to\infty} \| d\varphi_t|_{X^s(v)} \| = 0.$$

Lemma 3.2. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and bounded asymptote. Let $v \in S\mathcal{M}$ be a regular, recurrent vector and $\epsilon > 0$. Then there exist a neighborhood $U = U(v, \epsilon)$ of v in $\mathcal{R}$ and a real number $T = T(v, \epsilon)$ such that for all $w \in U$ and $\zeta \in X^s(v)$ with $\| \zeta \| = 1$,*

$$\| d_w(\varphi_t)\zeta \| \leq \epsilon \qquad \forall t \geq T.$$

Proof. By lemma 3.1 $\lim_{t\to\infty} \| d\varphi_t|_{X^s(v)} \| = 0$, so given $\epsilon_0 > 0$, there exists $T = T(\epsilon_0) > 0$ such that for all $t \geq T(\epsilon_0)$, $\| d\varphi_t|_{X^s(v)} \| \leq \frac{\epsilon_0}{2}$. By the continuity of the map $v \to X^s(v)$, there exists a neighborhood U of v such that $\| d\varphi_T|_{X^s(w)} \| \leq \epsilon_0$ for all $w \in U$.

By bounded asymptote, $\| d\varphi_t|_{X^s(w)} \| \leq C\epsilon_0 \quad \forall t \geq T \quad \forall w \in U$.

Now take $\epsilon_0 = \frac{\epsilon}{C}$. $\square$

Our goal is to prove that the Pesin set of the geodesic flow coincides a.e with $\mathcal{R}$.

Remark. Let v be a regular, recurrent vector and let ϵ be a real number such that $0 < \epsilon < 1$. Let U be the neighborhood of v and T the positive number given by lemma 3.2. Let μ be the normalized Lebesgue measure on $S\mathcal{M}$.

Let f_U be the characteristic function of U. Since $\varphi_T : S\mathcal{M} \to S\mathcal{M}$ is measure preserving, by Birkhoff's theorem [3], the orbital average $\hat{f}_U$ of f_U exists in a full measure set X_T and if $\hat{U} = \{w \in U \cap X_T; \quad \hat{f}_U(w) > 0\}$, then $\mu(\hat{U}) = \mu(U)$.

So we give the next lemma.

Lemma 3.3. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and bounded asymptote. Let v be a regular, recurrent vector, $0 < \epsilon < 1$ and let $U(v, \epsilon)$, and $T(v, \epsilon)$ be as in lemma 3.2. Then for all $w \in \hat{U}(v, \epsilon) \subset U(v, \epsilon) \cap X_T$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \| \mathrm{d}\varphi_t|_{X^s(w)} \| < 0.$$

Proof. Let $s_n = \#\{0 \le j \le n-1; \quad \varphi_{jT}(w) \in U\}$.
For $n \in \mathbb{N}$, let

$$\{j_k; \quad 0 \le j_k \le n-1, \quad \varphi_{j_k T}(w) \in U \quad 1 \le k \le s_n\}$$

be an increasing sequence of natural numbers.

Note that for t and r real numbers we have that

$$\| d\varphi_{t+r}|_{X^s(w)} \| \le \| d\varphi_t|_{X^s(\varphi_r w)} \| . \| d\varphi_r|_{X^s(w)} \| .$$

Then

$$\| d\varphi_{nT}|_{X^s(w)} \| \le \| d\varphi_{(n-j_{s_n})T}|_{X^s(\varphi_{j_{s_n}T}(w))} \| . \| d\varphi_{j_{s_n}T}|_{X^s(w)} \|$$

$$\| d\varphi_{j_{s_n}T}|_{X^s(w)} \| \le \| d\varphi_{(j_{s_n}-j_{s_n-1})T}|_{X^s\varphi_{j_{s_n-1}T}(w)} \| . \| d\varphi_{j_{s_n-1}T}|_{X^s(w)} \| .$$

And

$$\| d\varphi_{j_2T}|_{X^s(w)} \| \le \| d\varphi_{(j_2-j_1)T}|_{X^s(\varphi_{j_1T}(w))} \| . \| d\varphi_{j_1T}|_{X^s(w)} \| .$$

Then

$\| d\varphi_{nT}|_{X^s(w)} \| \le$

$\| d\varphi_{(n-j_{s_n})T}|_{X^s(\varphi_{j_{s_n}T}(w))} \| . \| d\varphi_{j_1T}|_{X^s(w)} \| . \prod_{k=2}^{s_n} \| d\varphi_{(j_k-j_{k-1})T}|_{X^s(\varphi_{(j_{k-1})T}(w))} \| .$

Since $j_k - j_{k-1} \ge 1$ and $\varphi_{j_{k_n}T}(w) \in U$ by hypothesis and bounded asymptote, we have that

$$\| d\varphi_{nT}|_{X^s(w)} \| \le C^2 \prod_{k=2}^{s_n} \epsilon = C^2 \epsilon^{s_n-1}.$$

And then

$$\lim_{n\to\infty} \frac{1}{n} \log \| d\varphi_{nT}|_{X^s(w)} \| \le \lim_{n\to\infty} \frac{1}{n}(s_n-1)\log\epsilon = \hat{f}_U(w)\log\epsilon < 0.$$

Therefore, we conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \log \| d\varphi_t|_{X^s(w)} \| < 0. \qquad \square$$

Corollary 3.1. *Let $\mathcal{M}$ be a compact Riemannian manifold with no conjugate points and bounded asymptote whose geodesic flow is expansive. Let $\mathcal{R}_0$ be the set of all regular vectors v such that*

$$\liminf_{t\to\infty} \frac{1}{t} \log \| \mathrm{d}\varphi_t|_{X^s(v)} \| < 0.$$

Then $\mu(\mathcal{R}_0) = \mu(\mathcal{R})$, where μ is the normalized Lebesgue measure on $S\mathcal{M}$.

Proof. By lemma 2.4 every regular vector v has $rank(v) = 1$. Now apply lemma 3.3 to get the corollary.

Proof of Theorem A. From Oseledec's theorem [13], there exists a total probability Borel set Λ_0 such that at each $v \in \Lambda_0$, $\lim_{t\to\infty} \frac{1}{t} \log \| d\varphi_t \zeta \|$ exists for all $\zeta \in T_v S\mathcal{M}$. And then, using Corollary 3.1 there exists a subset

$$\sum = \{v \in \mathcal{R}; \quad \lim_{t\to\pm\infty} \frac{1}{t} \log \| d\varphi_t \zeta \| < 0 \quad \forall \zeta \in X^s(v)\}$$

such that $\mu(\sum) = \mu(\mathcal{R})$.

If $S\colon S\mathcal{M} \to S\mathcal{M}$ is the map which takes a vector v into $-v$, then this map is an isometry such that $X^u(v) = dSX^s(-v)$ for all vector v.

Notice that $S(\Sigma) = \Sigma$ μ-a.e., since $S(\mathcal{R}) = \mathcal{R}$ (S preserves the behaviour of Jacobi fields) and hence

$$0 = \mu(\mathcal{R} - \Sigma) = \mu(S(\mathcal{R} - \Sigma)) = \mu(S(\mathcal{R}) - S(\Sigma)) = \mu(\mathcal{R} - S(\Sigma)).$$

Thus, Σ and $S(\Sigma)$ are subsets of $\mathcal{R}$ of total (relative) Lebesgue measure so they coincide up to a null measure subset of R.

Let $v \in \sum$ and $\eta \in X^u(v)$. Then there exists a vector $\zeta \in X^s(-v)$ such that $\eta = dS\zeta$. Since $S\varphi_t = \varphi_{-t}S$ for all $t \in \mathbb{R}$, we have that

$$d\varphi_t \eta = dSd\varphi_{-t}\zeta.$$

And since S is an isometry,

$$\| d\varphi_t \eta \| = \| d\varphi_{-t}\zeta \| .$$

Then

$$\lim_{t\to+\infty} \frac{1}{t} \log \| d\varphi_t \eta \| = -\lim_{t\to+\infty} \frac{1}{-t} \log \| d\varphi_{-t}\zeta \| .$$

But, $-v \in \sum$ and $\zeta \in X^s(-v)$, then

$$\lim_{t\to+\infty} \frac{1}{t} \log \| d\varphi_t \eta \| > 0.$$

Therefore, for all $v \in \sum$, $\zeta \in X^s(v)$ and $\eta \in X^u(v)$, the Lyapunov exponents $\mathcal{X}_v(\zeta)$ and $\mathcal{X}_v(\eta)$ are negative and positive respectively and then we conclude that $\sum \subset \Lambda$.

Hence we conclude that $\mu(\sum) \leq \mu(\Lambda) \leq \mu(\mathcal{R})$. That is,

$$\mu(\Lambda) = \mu(\mathcal{R}).$$

To finish the proof of theorem A, let us show that the set of closed hyperbolic orbits of the geodesic flow is dense in the unit tangent bundle.

Let v be a regular, periodic vector, let $U(v,\epsilon)$, $T(v,\epsilon)$ be given by lemma 3.2, without loss of generality we can suppose that T is a period of v and the orbital average $\hat{f}_U(v) > 0$. Applying lemma 3.3 and using the fact that v is a periodic vector, we have that there exist $A, B > 0$ such that

$$\| d\varphi_{nT}|_{X^s(v)} \| \leq Ae^{-nBT},$$

which implies immediately that the orbit of v is hyperbolic.

Let $V \in S\mathcal{M}$ be an arbitrary open set. Since the set of regular points is a dense set, then there exists a regular vector v such that $v \in V$. Let $\hat{V}$ be a small neighborhood of v such that $\hat{V} \subset V \cap \mathcal{R}$. Since the geodesic flow is expansive, the set of the periodic points is dense in $S\mathcal{M}$ (Ruggiero [18]). Then there exists a periodic vector $\hat{v} \in \hat{V} \subset V$. Since $\hat{v}$ is a regular periodic vector, we have that the closed orbit of $\hat{v}$ is hyperbolic. □

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